Some Considerations in the Theories of Combinations, Probabilities, and Life Contingencies. By PETER HARDY, ESQ., F.R.S. *and* F.I.A., *Actuary to the London Assurance Corporation.*

[Concluded from No. VI.]

BEFORE I proceed to the more immediate object of this paper, viz., to the Consideration of some leading Problems in the Theory of Probabilities, I would premise a few observations with reference to the doctrine of Life Contingencies.

In my work on Notation, p. 13, I have asserted a fact which is of such essential importance in the elucidation of the doctrine, that it can scarcely be too often repeated, or too strongly impressed upon the mind of a student, viz., that *curtate expectations of life are the values of annuities which do not bear interest;* and this is universally true, whether they be the expectations of single lives, of joint lives, of the longest of two or more lives, or of a life or lives after another or others,—in short, however remote or complicated the contingencies may be in which they are involved; and the values of annuities, or those series in which interest is involved, are invariably in the same form of expression as their corresponding curtate expectations, as thus employing the notation explained in the work already alluded to. The curtate expectation of the life, A, will be expressed by the series

$$\frac{a_1}{a} + \frac{a_2}{a} + \frac{a_3}{a} + \frac{a_4}{a} +, \text{\&c.}, = \mathrm{E}\mathrm{A},$$

and the value of an annuity of £1, during the same single life, will be expressed by the series

$$\frac{a_1}{ar} + \frac{a_2}{ar^2} + \frac{a_3}{ar^3} + \frac{a_4}{ar^4} + \text{\&c.}, = \mathrm{I}\mathrm{A};$$

now the series of which EA is the sum, differs only from the series of which IA is the sum, in the circumstance that interest is involved in the latter series as an element in the calculation; in like manner, the curtate expectation of the *joint* existence of the two lives, A and B, will be expressed by the series

$$\frac{a_1b_1}{ab} + \frac{a_2b_2}{ab} + \frac{a_3b_3}{ab} + \frac{a_4b_4}{ab} + \text{\&c.}, = \mathrm{E}\overline{\mathrm{AB}}^{1};$$

and the value of an annuity of £1, payable during the *joint* existence of the same two lives,—that is to say, payable until *one* drops, —will be expressed by the series

$$\frac{a_1b_1}{abr}+\frac{a_2b_2}{abr^2}+\frac{a_3b_3}{abr^3}+\frac{a_4b_4}{abr^4}+\&c., = \mathrm{I}\overline{\mathrm{AB}}^{1};$$

and, in like manner, the series of which $\mathrm{E}\overline{\mathrm{AB}}^{1}$ is the sum, differs only from the series of which $\mathrm{I}\overline{\mathrm{AB}}^{1}$ is the sum, in the circumstance that interest is involved in the latter series as an element in the calculation.

It will moreover be seen that, in each series, whether involving interest or not, all the terms subsequent to the first, are of a corresponding form of expression with such first term, so that such first term may be taken as indicating the character, and consequently the sum, of the series, as thus :—

$$\frac{a_1}{a}=\mathrm{E}\mathrm{A},$$

equal to the curtate expectation of the life of A ;

$$\frac{a_1}{ar}=\mathrm{I}\mathrm{A},$$

equal to the value of an annuity on the life of A ;

$$\frac{a_1b_1}{ab}=\mathrm{E}\overline{\mathrm{AB}}^{1},$$

equal to the curtate expectation of the joint lives, A and B ;

$$\frac{a_1b_1}{abr}=\mathrm{I}\overline{\mathrm{AB}}^{1},$$

equal to the value of an annuity on the joint lives A, and B.

Hence, if in any given case we have determined the form of expression for the expectation, we shall, in point of fact, have determined the form of expression for the corresponding annuity also.

Problem I.—The number of cases in favour of the *happening independently*, of three independent events, being

a_1 out of a,
b_1 out of b,
c_1 out of c,

required the *probabilities* of their happening separately ; or, in other words,—

To find the probability of an event happening *once* in *one* trial.

From what has been previously stated, it is manifest that the probability of the first event happening by itself will be $\frac{a_1}{a}$, which is also the probability that the first event will happen once in one trial.

The probability of the second event happening by itself will be $\frac{b_1}{b}$, which is also the probability that the second event will happen once in one trial.

The probability of the third event happening by itself will be $\frac{c_1}{c}$, which is also the probability that the third event will happen once in one trial. Now, let the separate events, which are to happen, be, that three given lives, A, B, and C, shall survive a certain assigned period of time,—say one year,—then we shall have $\frac{a_1}{a}$ for the probability or expectation that A will survive one year, and $\frac{b_1}{b}$ for the probability or expectation that B will survive one year, and $\frac{c_1}{c}$ for the probability or expectation that C will survive one year; consequently, as the terms subsequent to the first are similar in form to each first term,

$$\frac{a_1}{a} \text{ will indicate } \mathrm{E}_A,$$

equal to the curtate expectation of the life A;

$$\frac{b_1}{b} \text{ will indicate } \mathrm{E}_B,$$

equal to the curtate expectation of the life B;

$$\frac{c_1}{c} \text{ will indicate } \mathrm{E}_C,$$

equal to the curtate expectation of the life C.

PROBLEM II.—To find the probability of an event happening *once* in *two* trials.

The probability of the event happening in the first trial (according to the preceding problem) is $\frac{a_1}{a}$, which is the first part of the total probability required; but the event may fail in the first, and happen in the second trial, the probability of which is—

$$\left(1-\frac{a_1}{a}\right)\frac{a_1}{a}=\frac{a_1}{a}-\frac{a_1^2}{a^2},$$

which is the second part of the probability required; consequently, the sum of these two parts, or

$$\frac{a_1}{a}+\frac{a_1}{a}-\frac{a_1^2}{a^2}=\frac{2a_1}{a}-\frac{a_1^2}{a^2},$$

is the total required probability of the event happening once at least in two trials. Instead of the event being required to happen once in two trials, let the condition be that one at least out of two events shall happen, which is obviously the same with the preceding, only differently expressed, and let the new condition be, that out of two lives, A and B, one at least shall survive one year, it is manifest that in place of

$$\frac{2a_1}{a}-\frac{a_1^2}{a^2},$$

we shall have

$$\frac{a_1}{a}+\frac{b_1}{b}-\frac{a_1b_1}{ab},$$

equal to the probability required; and further substituting E_{A} for $\frac{a_1}{a}$, and E_{B} for $\frac{b_1}{b}$, &c., we shall have

$$\mathrm{E}_{\mathrm{A}}+\mathrm{E}_{\mathrm{B}}-\mathrm{E}_{\overline{\mathrm{AB}}}^{\,1},$$

which is the expression for the expectation of the longest of two lives.*

Problem III.—To find the probability of an event happening *once* in *three* trials.

It may happen in the first trial, the probability of which is $\frac{a_1}{a}$, and is the first part of the probability required; but it may fail in the first trial, and happen, *once at least,* in the two remaining trials; this probability, according to Problem II., is $\frac{2a_1}{a}-\frac{a_1^2}{a^2}$, therefore

$$\left(1-\frac{a_1}{a}\right)\times\left(\frac{2a_1}{a}-\frac{a_1^2}{a^2}\right)$$

is equal to

$$\frac{2a_1}{a}-\frac{3a_1^2}{a^2}+\frac{a_1^3}{a^3},$$

* See Baily, chap. ii., Prob. 2, p. 50; and Hardy's *Notation*, p. 33.

which is the second part of the required probability; the sum of these two parts, or

$$\frac{a_1}{a}+\frac{2a_1}{a}-\frac{3a_1^2}{a^2}+\frac{a_1^3}{a^3}=\frac{3a_1}{a}-\frac{3a_1^2}{a^2}+\frac{a_1^3}{a^3},$$

which is the total probability that an event will happen once at least in three trials. Now, as before, let us substitute separate events for separate trials, and let those events be again the lives A, B, and C, and, instead of the formula immediately foregoing, we shall obtain

$$\frac{a_1}{a}+\frac{b_1}{b}+\frac{c_1}{c}-\frac{a_1b_1}{ab}-\frac{a_1c_1}{ac}-\frac{b_1c_1}{bc}+\frac{a_1b_1c_1}{abc};$$

which may, as in the preceding problem, be represented by

$$\mathrm{E}\mathrm{A}+\mathrm{E}\mathrm{B}+\mathrm{E}\mathrm{C}-\mathrm{E}\overline{\mathrm{AB}}^{1}-\mathrm{E}\overline{\mathrm{AC}}^{1}-\mathrm{E}\overline{\mathrm{BC}}^{1}+\mathrm{E}\overline{\mathrm{ABC}}^{1},$$

which formula is the expression for the expectation of the longest of three lives.*

Problem IV.—To find the probability of an event happening *twice* in *two* trials.

This is manifestly equal to the probability of the event happening twice in succession, or, what is the same thing, two separate events happening together, which is equal to the product of the separate probabilities—

$$\frac{a_1}{a}\times\frac{a_1}{a}=\frac{a_1^2}{a^2};$$

substituting lives as before, we obtain

$$\frac{a_1}{a}\times\frac{b_1}{b}=\frac{a_1b_1}{ab};\text{ and, consequently, }\mathrm{E}\overline{\mathrm{AB}}^{1},$$

which is the expression for the expectation of the two joint lives, A and B.

Problem V.—To find the probability of an event happening *twice* in *three* trials.

The event may happen once in the first trial, equal to $\frac{a_1}{a}$; and once also in the two succeeding trials, equal to $\frac{2a_1}{a}-\frac{a_1^2}{a^2}$; the product of these two quantities is equal to

* Baily, Prob. 2, p. 50; Hardy's *Notation*, p. 36.

$$\frac{a_1}{a}\left(\frac{2a_1}{a}-\frac{a_1^2}{a^2}\right)=\frac{2a_1^2}{a^2}-\frac{a_1^3}{a^3},$$

and is equal to the first part of the required probability. But the event may fail in the *first* trial, and happen successively in the two last, equal to the product of

$$\left(1-\frac{a_1}{a}\right)\frac{a_1^2}{a^2}=\frac{a_1^2}{a^2}-\frac{a_1^3}{a^3},$$

which will be the second part of the required probability; and the sum of these two parts, viz.,

$$\frac{2a_1^2}{a^2}-\frac{a_1^3}{a^3}+\frac{a_1^2}{a^2}-\frac{a_1^3}{a^3}=\frac{3a_1^2}{a^2}-\frac{2a_1^3}{a^3},$$

will be equal to the total probability required. Now, substituting, as before, the three lives A, B, and C, we obtain

$$\frac{a_1b_1}{ab}+\frac{a_1c_1}{ac}+\frac{b_1c_1}{bc}-\frac{2\,a_1b_1c_1}{abc},$$

equivalent to

$$\mathrm{E}_{\overline{\mathrm{AB}}}^{\,1}+\mathrm{E}_{\overline{\mathrm{AC}}}^{\,1}+\mathrm{E}_{\overline{\mathrm{BC}}}^{\,1}-2\,\mathrm{E}_{\overline{\mathrm{ABC}}}^{\,1},$$

which is the expression for the expectation of the *longest of any two* out of three lives.*

Problem VI.—To find the probability of an event happening *three* times in *three* trials.

Here we have

$$\frac{a_1}{a}\times\frac{a_1}{a}\times\frac{a_1}{a}=\frac{a_1^3}{a^3}$$

for the probability required, and substituting the three lives, A, B, and C, as in the preceding problems, we obtain

$$\frac{a_1b_1c_1}{abc},\ \text{equivalent to}\ \mathrm{E}_{\overline{\mathrm{ABC}}}^{\,1},$$

or to the expectation of the three joint lives.

Problem VII.—To find the probability of an event happening *once*, and *once only*, in *two* trials.

This problem must be carefully distinguished from the second problem, in which it was required that the event should happen once at least in two trials, there being no restriction placed on its

* Baily, chap. iii. p. 58; Hardy's *Notation*, p. 36.

happening *more than once;* but here the problem is restricted to the event happening *once,* and *once only,* implying that such event is to happen in one trial and to fail in the other, as thus, it may *happen* in the *first* trial, and *fail* in the second; equal to

$$\frac{a_1}{a}\left(1-\frac{a_1}{a}\right)=\frac{a_1}{a}-\frac{a_1^2}{a^2},$$

which is the first part of the required probability; or, *vice versâ,* it may *fail* in the *first* trial and happen in the second; equal to

$$\left(1-\frac{a_1}{a}\right)\frac{a_1}{a}=\frac{a_1}{a}-\frac{a_1^2}{a^2},$$

which is the second part of the required probability; and the sum of these two parts will be

$$\frac{2a_1}{a}-\frac{2a_1^2}{a^2},$$

equal to the total probability required. Now, if we substitute two lives, A and B, as in former instances, we shall, instead of the expression $\frac{2a_1}{a}-\frac{2a_1^2}{a^2}$, obtain

$$\frac{a_1}{a}+\frac{b_1}{b}-\frac{2a_1b_1}{ab},$$

which, as before, is equivalent to

$$\mathrm{E}_{\mathrm{A}}+\mathrm{E}_{\mathrm{B}}-2\mathrm{E}_{\overset{1}{\overline{\mathrm{AB}}}},$$

and is the expression for the *reversionary expectation* of the life of the survivor, after the extinction of the first of the two joint lives A and B.*

Problem VIII.—To find the probability of an event happening *once,* and *once only,* in *three* trials.

It may happen in the first trial, and fail in the second and third trials; equal to

$$\frac{a_1}{a}\left(1-\frac{a_1}{a}\right)\times\left(1-\frac{a_1}{a}\right)=$$

$$\frac{a_1}{a}\left(1-\frac{a_1}{a}-\frac{a_1}{a}+\frac{a_1^2}{a^2}\right)=\frac{a_1}{a}-\frac{2a_1^2}{a^2}+\frac{a_1^3}{a^3},$$

equal to the first part of the total probability required. But it

* Baily, chap. iii., and Hardy's *Notation*, p. 33.

may fail in the first trial, and happen once, and once only, in the second and third trials (Prob. VII.)—

$$\left(1-\frac{a_1}{a}\right)\times\left(\frac{2a_1}{a}-\frac{2a_1^2}{a^2}\right)=\frac{2a_1}{a}-\frac{4a_1^2}{a^2}+\frac{2a_1^3}{a^3},$$

equal to the second part of the required probability; the sum of the first and second parts will be equal to

$$\frac{3a_1}{a}-\frac{6a_1^2}{a^2}+\frac{3a_1^3}{a^3},$$

which is the total probability required of an event happening once, and once only, in three trials.

If we now substitute, as before, the three lives A, B, and C, we shall obtain the corresponding expression—

$$\frac{a_1}{a}+\frac{b_1}{b}+\frac{c_1}{c}-\frac{2a_1b_1}{ab}-\frac{2a_1c_1}{ac}-\frac{2b_1c_1}{bc}+\frac{3a_1b_1c_1}{abc},$$

equivalent to

$$\mathrm{E}_{\mathrm{A}}+\mathrm{E}_{\mathrm{B}}+\mathrm{E}_{\mathrm{C}}-2\mathrm{E}_{\overline{\mathrm{AB}}}^{\,1}-2\mathrm{E}_{\overline{\mathrm{AC}}}^{\,1}-2\mathrm{E}_{\overline{\mathrm{BC}}}^{\,1}+3\mathrm{E}_{\overline{\mathrm{ABC}}}^{\,1},$$

which is the expression for the reversionary expectation of the life of the last survivor out of the three lives, A, B, and C, after the extinction of the other two, which is manifestly equal to the final expression arrived at in Problem III., lessened by that arrived at in Problem V.*

PROBLEM IX.—To find the probability of an event happening *twice*, and *twice only*, in *three* trials.

Here the event may happen once in the first trial, and may happen once again, but once only, in the last two trials, as in Problem VII., equal to

$$\frac{a_1}{a}\left(\frac{2a_1}{a}-\frac{2a_1^2}{a^2}\right)=\frac{2a_1^2}{a^2}-\frac{2a_1^3}{a^3},$$

to be reserved, as the first part of the probability required. But the event may fail in the first trial, and happen successively in the two remaining trials, equal to

$$\left(1-\frac{a_1}{a}\right)\frac{a_1^2}{a^2}=\frac{a_1^2}{a^2}-\frac{a_1^3}{a^3},$$

* The above expression is obviously identical with $\mathrm{E}_{\overline{\mathrm{ABC}}}^{\,3}-\mathrm{E}_{\overline{\mathrm{ABC}}}^{\,2}$.—See Hardy's *Notation*, p. 38.

for the second part of the probability required; the sum of these two parts,

$$\frac{3a_1^2}{a^2} - \frac{3a_1^3}{a^3},$$

will be equal to the total probability of an event happening twice, and twice only, in three trials; substituting the three lives, A, B, and C, as before, we obtain

$$\frac{a_1b_1}{ab} + \frac{a_1c_1}{ac} + \frac{b_1c_1}{bc} - \frac{3a_1b_1c_1}{abc},$$

or its corresponding expectation,

$$\mathrm{E}\overset{1}{\overline{\mathrm{AB}}} + \mathrm{E}\overset{1}{\overline{\mathrm{AC}}} + \mathrm{E}\overset{1}{\overline{\mathrm{BC}}} - 3\mathrm{E}\overset{1}{\overline{\mathrm{ABC}}},$$

equal to the reversionary expectation of the *joint* existence of any two survivors out of the three lives, A, B, and C; or, what is the same thing, equal to the expectation arrived at in Problem V., lessened by the expectation arrived at in Problem VI.*

The foregoing nine problems nearly exhaust the subject, so far as three lives only are involved, without having reference to any particular order of survivorship; but, before concluding this paper, I would again invite attention to the doctrine of combinations. It has been already shown that the horizontal columns, both in the alphabetical and in the numeral tables given in pages 155 and 157, express the coefficients of an expanded binomial; the rationale of this will be sufficiently obvious to all my readers who are sufficiently advanced in analysis to demonstrate the binomial theorem for affirmative and integral powers, inasmuch as the constant multiplication of a and b into each other, and into their successive powers, gives rise to a perfect series of combinations.

The immediate application of the law of combinations to that of probabilities will be made more readily apparent by arranging the foregoing problems, and a few additional and general cases, into tabular forms, in the manner following.

* The equivalent expression for the formula given in the text is $\mathrm{E}\overset{2}{\overline{\mathrm{ABC}}} - \mathrm{E}\overset{1}{\overline{\mathrm{ABC}}}$. —See Hardy's *Notation*, p. 37.

TABLE I.

When the event is to happen	The probability is
Once in one trial	$\frac{a_1}{a}$
Once in two trials	$\frac{2a_1}{a} - \frac{a_1^2}{a^2}$
Once in three trials	$\frac{3a_1}{a} - \frac{3a_1^2}{a^2} + \frac{a_1^3}{a^3}$
Once in four trials	$\frac{4a_1}{a} - \frac{6a_1^2}{a^2} + \frac{4a_1^3}{a^3} - \frac{a_1^4}{a^4}$
Once in five trials	$\frac{5a_1}{a} - \frac{10a_1^2}{a^2} + \frac{10a_1^3}{a^3} - \frac{5a_1^4}{a^4} + \frac{a_1^5}{a^5}$

TABLE II.

When the event is to happen	The probability is
Twice in two trials	$\frac{a_1^2}{a^2}$
Twice in three trials	$\frac{3a_1^2}{a^2} - \frac{2a_1^3}{a^3}$
Twice in four trials	$\frac{6a_1^2}{a^2} - \frac{8a_1^3}{a^3} + \frac{3a_1^4}{a^4}$
Twice in five trials	$\frac{10a_1^2}{a^2} - \frac{20a_1^3}{a^3} + \frac{15a_1^4}{a^4} - \frac{4a_1^5}{a^5}$
Twice in six trials......	$\frac{15a_1^2}{a^2} - \frac{40a_1^3}{a^3} + \frac{45a_1^4}{a^4} - \frac{24a_1^5}{a^5} + \frac{5a_1^6}{a^6}$

TABLE III.

When the event is to happen	The probability is
Once only in one trial . . .	$\frac{a_1}{a}$
Once only in two trials . .	$\frac{2a_1}{a} - \frac{2a_1^2}{a^2}$
Once only in three trials .	$\frac{3a_1}{a} - \frac{6a_1^2}{a^2} + \frac{3a_1^3}{a^3}$
Once only in four trials . .	$\frac{4a_1}{a} - \frac{12a_1^2}{a^2} + \frac{12a_1^3}{a^3} - \frac{4a_1^4}{a^4}$
Once only in five trials . .	$\frac{5a_1}{a} - \frac{20a_1^2}{a^2} + \frac{30a_1^3}{a^3} - \frac{20a_1^4}{a^4} + \frac{5a_1^5}{a^5}$

TABLE IV.

When the event is to happen	The probability is
Twice only in two trials .	$\frac{a_1^2}{a^2}$
Twice only in three trials	$\frac{3a_1^2}{a^2} - \frac{3a_1^3}{a^3}$
Twice only in four trials .	$\frac{6a_1^2}{a^2} - \frac{12a_1^3}{a^3} + \frac{6a_1^4}{a^4}$
Twice only in five trials . .	$\frac{10a_1^2}{a^2} - \frac{30a_1^3}{a^3} + \frac{30a_1^4}{a^4} - \frac{10a}{a^5}$
Twice only in six trials . .	$\frac{15a_1^2}{a^2} - \frac{60a_1^3}{a^3} + \frac{90a_1^4}{a^4} - \frac{60a_1^5}{a^5} + \frac{15a_1^6}{a^6}$

Table V.

When the event is to happen	The probability is
Once in n trials....................	$n \cdot \frac{a_1}{a} - \frac{n.\overline{n-1}}{2} \frac{a_1^2}{a^2} + \frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{a_1^3}{a^3} -$, &c.
Twice in n trials	$\frac{n.\overline{n-1}}{2} \frac{a_1^2}{a^2} - \frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{2a_1^3}{a^3} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} \frac{3a_1^4}{a^4}$, &c.
Thrice in n trials	$\frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{a_1^3}{a^3} - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} \frac{3a_1^4}{a^4} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5} \frac{6a_1^5}{a^5}$, &c.
And generally m times in n trials....	$\frac{n.\overline{n-1}.\overline{n-2}\ldots\overline{n-m-1}}{2.3\ldots m} \frac{a_1^m}{a^m} - \frac{n.\overline{n-1}\ldots\overline{n-m}}{2.3..m+1} . m \frac{a_1^{m+1}}{a^{m+1}} + \frac{n.\overline{n-1}\ldots\overline{n-m+1}}{2.3\ldots m+2} \frac{m.\overline{m+1}}{2} \frac{a_1^{m+2}}{a^{m+2}}$, &c.

Table VI.

When the event is to happen,	The probability is
Once only in n trials..............	$n \cdot \frac{a_1}{a} - \frac{n.\overline{n-1}}{2} \frac{2a_1^2}{a^2} + \frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{3a_1^3}{a^3} -$, &c.
Twice only in n trials	$\frac{n.\overline{n-1}}{2} \frac{a_1^2}{a^2} - \frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{3a_1^3}{a^3} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} \frac{6a_1^4}{a^4}$, &c.
Thrice only in n trials	$\frac{n.\overline{n-1}.\overline{n-2}}{2.3} \frac{a_1^3}{a^3} - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} \frac{4a_1^4}{a^4} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5} \frac{10a_1^5}{a^5}$, &c.
And generally m times only in n trials	$\frac{n.\overline{n-1}\ldots\overline{n.\overline{n-1}}}{2.3.m} \frac{a_1^m}{a^m} - \frac{n.\overline{n-1}\ldots\overline{n-m}}{2.3..m+1} . \overline{m+1} . \frac{a^{m+1}}{a^{m+1}} + \frac{n.\overline{n-1}\ldots\overline{n-m+1}}{2.3\ldots m+2} \frac{\overline{m+1}.\overline{m+2}}{2} \frac{a_1^{m+2}}{a^{m+2}}$, &c.